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Asymptotic distributions for a class of generalized L -statistics

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We adapt the techniques in Stigler [*Ann. Statist.* **1** (1973) 472–477] to obtain a new, general asymptotic result for trimmed U -statistics via the generalized L -statistic representation introduced by Serfling [*Ann. Statist.* **12** (1984) 76–86]. Unlike existing results, we do not require continuity of an associated distribution at the truncation points. Our results are quite general and are expressed in terms of the quantile function associated with the distribution of the U -statistic summands. This approach leads to improved conditions for the asymptotic normality of these trimmed U -statistics.

Keywords: generalized L -statistics; trimmed U -statistics; U -statistics; weak convergence

1. Introduction and statement of results

Stigler [23] developed an asymptotic result for the trimmed mean without requiring continuity of the underlying distribution function associated with the observations. This result was extended to non-degenerate U -statistics based on trimmed samples in Borovskikh and Weber [4]. An alternative method for developing robust versions of U -statistics is to consider the statistic formed by trimming the kernel values, rather than the observations upon which the statistic is based. This idea is discussed in, for example, Serfling [18], Choudhury and Serfling [7] and Gijbels, Janssen and Veraverbeke [10]. In this paper, we use the generalized L -statistic representation developed in Serfling [18] to obtain an asymptotic result for trimmed U -statistics under quite general conditions. We will not require continuity of the relevant, associated distribution at the truncation points.

Let X, X_1, \dots, X_n be independent identically distributed random variables, taking values in a measurable space $(X, \mathcal{B}(X))$ and having common distribution F . Let h be a symmetric function from X^m to \mathcal{R} and denote by H_F the right-continuous distribution function of the random variable $h(X_1, \dots, X_m)$. Set $N = \binom{n}{m}$ and let h_1, \dots, h_N be an enumeration of the values of $h(X_{i_1}, \dots, X_{i_m})$ taken over the N m -tuples in

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$\sigma_{nm} = \{(i_1, \dots, i_m) : 1 \leq i_1 < \dots < i_m \leq n\}$. Note that these random variables h_i are, in general, dependent. Let $h_{n1} \leq \dots \leq h_{nN}$ denote the ordered values of h_1, \dots, h_N .

The original U -statistic is defined as an average taken over the N possible outcomes $h(X_{i_1}, \dots, X_{i_m}), 1 \leq i_1 < \dots < i_m \leq n$, that is,

$$U = \binom{n}{m}^{-1} \sum_{\sigma_{nm}} h(X_{i_1}, \dots, X_{i_m}) = N^{-1} \sum_{i=1}^N h_{ni} = \int_R x dH_n(x), \quad (1)$$

where the empirical distribution function $H_n(x)$ of U -statistical structure is defined by

$$H_n(y) = \binom{n}{m}^{-1} \sum_{\sigma_{nm}} I\{h(X_{i_1}, \dots, X_{i_m}) \leq y\}, \quad y \in R, \quad (2)$$

and $I\{A\}$ denotes the indicator of the set A . For any $0 < \gamma < 1$, let $N_\gamma = [\gamma N]$, where $[a]$ denotes the largest integer less than or equal to a . If $0 < \alpha < \beta < 1$, then put $N_{\alpha\beta} = N_\beta - N_\alpha$. The trimmed versions of U are based on trimming the second sum in (1),

$$U_{\alpha\beta} = N_{\alpha\beta}^{-1} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni}, \quad (3)$$

or on trimming of the range of integration in (1),

$$L_{\alpha\beta} = \int_{[h_\alpha, h_\beta)} x dH_n(x), \quad (4)$$

with $h_\alpha = h_{n\bar{N}_\alpha}$ and $h_\beta = h_{n\bar{N}_\beta}$, where $\bar{N}_\gamma = -[-\gamma N], \gamma = \alpha, \beta$. For the results that follow, it is important to note that the lower bound for the integral in (4) is included and the upper bound excluded. This is critical since H_n is a step function. With this constraint, we are able to obtain the asymptotic distribution of $L_{\alpha\beta}$ without imposing any conditions on the nature of H_F . In Lemma 2.3, we show that $L_{\alpha\beta} = N^{-1} \sum_{i=\bar{N}_\alpha}^{\bar{N}_\beta-1} h_{ni}$. Thus, $U_{\alpha\beta}$ and $L_{\alpha\beta}$ differ in terms of their divisors, and there are possible subtle differences in the number of summands.

A class of generalized L -statistics, which includes (3) and (4), was introduced by Serfling [18]. The trimmed U -statistics (3) and (4) are directly connected with generalized Lorenz curves, which are important in financial mathematics (see, for example, Goldie [9], Helmers and Zitikis [13]).

Clearly, $H_n(y)$ is an unbiased estimator of $H_F(y)$. In the case $m = 1$ and $h(x) = x$, H_n reduces to the usual empirical distribution function. Define the left-continuous quantile function $H_F^{-1}(t) = \inf\{y \in R : H_F(y) \geq t\}, 0 < t \leq 1, H_F^{-1}(0) = H_F^{-1}(0+)$, for any distribution function H_F . The empirical quantile function $H_n^{-1}(t)$ has the form

$$H_n^{-1}(t) = \sum_{i=1}^N h_{ni} I\left\{\frac{i-1}{N} < t \leq \frac{i}{N}\right\}, \quad H_n^{-1}(0) = h_{n1}.$$

A large number of authors have studied the weak convergence of such L -statistics in the case $m = 1, h(x) = x$. A partial list consists of Chernoff *et al.* [6], Bickel [2], Shorack [19, 20], Stigler [23, 24], Csörgö *et al.* [8], Griffin and Pruitt [11], Cheng [5], Mason and Shorack [16]. For $m \geq 2$, under various sets of regularity conditions, asymptotic normality of various types of generalized L -statistics has been investigated by Silverman [21], Serfling [18], Akritas [1], Janssen *et al.* [15], Helmers and Ruymgaart [12], Gijbels *et al.* [10] and Hössjer [14].

In the aforementioned papers, for $m \geq 2$, the results always assumed that H_F is continuous or smooth. However, in modern statistical robust procedures and for bootstrap procedures, results allowing for the discontinuity of the underlying distribution function H_F are needed. We study the asymptotic behavior of $U_{\alpha\beta}$ and $L_{\alpha\beta}$ for any H_F without imposing the requirement of continuity.

The conditions of our theorem and the limit random variable are defined via the values of quantile function H_F^{-1} at the points α and β . Existing results handle the cases where $H_F^{-1}(\gamma+) = H_F^{-1}(\gamma), \gamma = \alpha, \beta$. Our main result is derived without this assumption of continuity. We represent the trimmed U -statistic as a sum of classical U -statistics with bounded, non-degenerate kernels plus some smaller terms and then we apply standard results to such statistics.

For convenience, in what follows, for the distribution function H_F , we denote the smallest quantile $H_F^{-1}(\gamma)$ and the largest quantile $H_F^{-1}(\gamma+)$ as, respectively,

$$\begin{aligned}\xi_\gamma^- &:= \inf\{x \in R : H_F(x) \geq \gamma\}, \\ \xi_\gamma^+ &:= \sup\{x \in R : H_F(x) \leq \gamma\}\end{aligned}$$

and $\Delta\xi_\gamma = \xi_\gamma^+ - \xi_\gamma^-$ with $\gamma = \alpha, \beta$. Let

$$\dot{N}_\gamma^\pm = \sum_{i=1}^N I\{h_i < \xi_\gamma^\pm\}, \quad N_\gamma^\pm = \sum_{i=1}^N I\{h_i \leq \xi_\gamma^\pm\}.$$

Note that

$$H_n(\xi_\gamma^\pm-) = N^{-1}\dot{N}_\gamma^\pm, \quad H_n(\xi_\gamma^\pm) = N^{-1}N_\gamma^\pm \quad (5)$$

and $H_n^{-1}(\gamma) = h_{n\dot{N}_\gamma}$ are valid for all $0 < \gamma < 1$ and the following events coincide:

$$\{H_n^{-1}(\gamma) > x\} = \{\gamma > H_n(x)\}, \quad \{H_n^{-1}(\gamma) \leq x\} = \{\gamma \leq H_n(x)\}, \quad x \in R. \quad (6)$$

Introduce the functional $\theta = \theta(H_F)$, where

$$\theta = \int_R [((x - \xi_\beta^-)I\{x \leq \xi_\beta^-\} + \beta\xi_\beta^-) - ((x - \xi_\alpha^+)I\{x < \xi_\alpha^+\} + \alpha\xi_\alpha^+)] dH_F(x)$$

and the following functions with $x \in \mathbb{X}$:

$$g(x) = [EI\{h(x, X_2, \dots, X_m) \leq \xi_\beta^-\}(h(x, X_2, \dots, X_m) - \xi_\beta^-) + \beta\xi_\beta^-]$$

$$\begin{aligned}
& - [EI\{h(x, X_2, \dots, X_m) < \xi_\alpha^+\}(h(x, X_2, \dots, X_m) - \xi_\alpha^+) + \alpha\xi_\alpha^+] - \theta, \\
g_\alpha(x) &= EI\{h(x, X_2, \dots, X_m) < \xi_\alpha^+\} - \theta_\alpha, \quad \theta_\alpha = H_F(\xi_\alpha^+), \\
g_\beta(x) &= EI\{h(x, X_2, \dots, X_m) \leq \xi_\beta^-\} - \theta_\beta \\
&= 1 - \theta_\beta - EI\{h(x, X_2, \dots, X_m) > \xi_\beta^-\}, \quad \theta_\beta = H_F(\xi_\beta^-).
\end{aligned} \tag{7}$$

Note that for all $0 < \alpha < \beta < 1$ and $x \in \mathbf{X}$, we have $|g(x)| \leq 4(|\xi_\alpha^+| + |\xi_\beta^-|)$.

Let $\sigma_g^2 = Eg^2(X)$, $\sigma_{g_\alpha}^2 = Eg_\alpha^2(X)$, $\sigma_{g_\beta}^2 = Eg_\beta^2(X)$, $c_{gg_\alpha} = Eg(X)g_\alpha(X)$, $c_{gg_\beta} = Eg(X)g_\beta(X)$ and $c_{g_\alpha g_\beta} = Eg_\alpha(X)g_\beta(X)$.

Theorem 1.1. *If $\sigma_g^2 > 0$, then for any underlying distribution function H_F , we have*

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g - \Delta\xi_\alpha I(\tau_\alpha > 0)\tau_\alpha - \Delta\xi_\beta I(\tau_\beta < 0)\tau_\beta,$$

where $(\tau_\alpha, \tau_g, \tau_\beta)$ is a trivariate Gaussian random vector with mean vector zero and covariance matrix

$$\begin{pmatrix} \sigma_{g_\alpha}^2 & c_{gg_\alpha} & c_{g_\alpha g_\beta} \\ c_{gg_\alpha} & \sigma_g^2 & c_{gg_\beta} \\ c_{g_\alpha g_\beta} & c_{gg_\beta} & \sigma_{g_\beta}^2 \end{pmatrix}.$$

Corollary 1.2. *For any underlying distribution function H_F , we have, when $\sigma_g^2 > 0$,*

$$\frac{\sqrt{n}}{m}(L_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g - \Delta\xi_\alpha I(\tau_\alpha > 0)\tau_\alpha - \Delta\xi_\beta I(\tau_\beta < 0)\tau_\beta,$$

where $(\tau_\alpha, \tau_g, \tau_\beta)$ is a trivariate Gaussian random vector defined as in Theorem 1.1.

Corollary 1.3. *Suppose that the quantile function $H_F^{-1}(x)$ is continuous at the points α and β . If $\sigma_g^2 > 0$, then*

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g.$$

For the simple case $m = 1$, the functions in (7) reduce to

$$\begin{aligned}
g(x) &= I\{\xi_\alpha^+ \leq h(x) \leq \xi_\beta^-\}h(x) - EI\{\xi_\alpha^+ \leq h(X) \leq \xi_\beta^-\}h(X) \\
&\quad + \xi_\alpha^+ g_\alpha(x) - \xi_\beta^- g_\beta(x), \\
g_\alpha(x) &= I\{h(x) < \xi_\alpha^+\} - \theta_\alpha, \\
g_\beta(x) &= I\{h(x) \leq \xi_\beta^-\} - \theta_\beta = 1 - \theta_\beta - I\{h(x) > \xi_\beta^-\}.
\end{aligned}$$

A useful application of the theorem for the $m = 2$ case is for the kernel $h(x, y) = \frac{1}{2}(x - y)^2$. This provides the asymptotic behavior of a natural, alternative robust version of

the sample variance. We will now develop explicit expressions for the terms in a more interesting example.

Example. Let $h(x_1, \dots, x_m) = \max\{x_1, \dots, x_m\}$ with $m \geq 2$. Let $F(t)$ be the distribution function of X_i and let $Y = \max\{X_2, \dots, X_m\}$. Then $H_F(t) = (F(t))^m$ and

$$\begin{aligned} g(x) &= g_{\alpha\beta}(x) + \xi_{\alpha}^{+} g_{\alpha}(x) - \xi_{\beta}^{-} g_{\beta}(x), \\ g_{\alpha\beta}(x) &= EI\{\xi_{\alpha}^{+} \leq \max\{x, Y\} \leq \xi_{\beta}^{-}\} \max\{x, Y\} - \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y dH_F(y) \\ &= I\{\xi_{\alpha}^{+} \leq x \leq \xi_{\beta}^{-}\} x(F(x))^{m-1} - \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y(F(y))^{m-1} dF(y) \\ &\quad + \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} (I\{x < y\} - F(y-)) y d(F(y))^{m-1}, \\ g_{\alpha}(x) &= I\{x < \xi_{\alpha}^{+}\} (F(\xi_{\alpha}^{+}-))^{m-1} - (F(\xi_{\alpha}^{+}-))^m, \\ g_{\beta}(x) &= I\{x \leq \xi_{\beta}^{-}\} (F(\xi_{\beta}^{-}))^{m-1} - (F(\xi_{\beta}^{-}))^m. \end{aligned}$$

In addition,

$$\begin{aligned} \sigma_g^2 &= Eg_{\alpha\beta}^2(X) + E[\xi_{\alpha}^{+} g_{\alpha}(X) - \xi_{\beta}^{-} g_{\beta}(X)]^2 \\ &\quad + 2Eg_{\alpha\beta}(X)[\xi_{\alpha}^{+} g_{\alpha}(X) - \xi_{\beta}^{-} g_{\beta}(X)], \\ \sigma_{g_{\alpha}}^2 &= (F(\xi_{\alpha}^{+}-))^{2m-1} (1 - F(\xi_{\alpha}^{+}-)), \quad \sigma_{g_{\beta}}^2 = (F(\xi_{\beta}^{-}))^{2m-1} (1 - F(\xi_{\beta}^{-})), \\ Eg_{\alpha}(X)g_{\beta}(X) &= (F(\xi_{\alpha}^{+}-)F(\xi_{\beta}^{-}))^{m-1} F(\xi_{\alpha}^{+}-)(1 - F(\xi_{\beta}^{-})), \\ Eg_{\alpha\beta}(X)g_{\alpha}(X) &= (F(\xi_{\alpha}^{+}-))^m \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} (1 - F(y-)) y d(F(y))^{m-1} \\ &\quad - (F(\xi_{\alpha}^{+}-))^m \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y(F(y))^{m-1} dF(y), \\ Eg_{\alpha\beta}(X)g_{\beta}(X) &= (F(\xi_{\beta}^{-}))^{m-1} (1 - F(\xi_{\beta}^{-})) \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} y(F(y))^{m-1} dF(y) \\ &\quad + (F(\xi_{\beta}^{-}))^{m-1} (1 - F(\xi_{\beta}^{-})) \int_{[\xi_{\alpha}^{+}, \xi_{\beta}^{-}]} F(y-) y d(F(y))^{m-1}. \end{aligned}$$

Consider the distribution function

$$\begin{aligned} F(t) &= 2tI\{0 \leq t < \tfrac{1}{2}\alpha^{1/m}\} + \alpha^{1/m}I\{\tfrac{1}{2}\alpha^{1/m} \leq t < \alpha^{1/m}\} \\ &\quad + tI\{\alpha^{1/m} \leq t < \beta^{1/m}\} + \beta^{1/m}I\{\beta^{1/m} \leq t < 2\beta^{1/m}\} \\ &\quad + \tfrac{1}{2}tI\{2\beta^{1/m} \leq t < 2\} + I\{t \geq 2\}, \quad t \in R. \end{aligned}$$

Then

$$\begin{aligned}\xi_{\alpha}^{-} &= \frac{1}{2}\alpha^{1/m}, & \xi_{\alpha}^{+} &= \alpha^{1/m}, & \xi_{\beta}^{-} &= \beta^{1/m}, & \xi_{\beta}^{+} &= 2\beta^{1/m}, \\ F(\xi_{\alpha}^{+}-) &= \alpha^{1/m}, & F(\xi_{\beta}^{-}) &= \beta^{1/m}, & F(t) &= t, & t &\in [\alpha^{1/m}, \beta^{1/m}], & \sigma_g^2 > 0\end{aligned}$$

and the limiting behavior is given by

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g - \frac{1}{2}\alpha^{1/m}I(\tau_{\alpha} > 0)\tau_{\alpha} - \beta^{1/m}I(\tau_{\beta} < 0)\tau_{\beta}.$$

However, for the simpler distribution function

$$F(t) = tI\{0 \leq t < 1\} + I\{t \geq 1\}, \quad t \in R,$$

we have

$$\begin{aligned}\xi_{\alpha}^{-} &= \xi_{\alpha}^{+} = \alpha^{1/m}, & \xi_{\beta}^{-} &= \xi_{\beta}^{+} = \beta^{1/m}, \\ F(\xi_{\alpha}^{+}-) &= \alpha^{1/m}, & F(\xi_{\beta}^{-}) &= \beta^{1/m}, & F(t) &= t, & t &\in [\alpha^{1/m}, \beta^{1/m}], & \sigma_g^2 > 0\end{aligned}$$

and we get the asymptotic behavior covered by Janssen *et al.* [15],

$$\frac{\beta - \alpha}{m} \sqrt{n}(U_{\alpha\beta} - \theta) \xrightarrow{d} \tau_g.$$

2. Proofs

The following two lemmas are key results for the proof.

Lemma 2.1. *The following representation holds:*

$$\begin{aligned}\sum_{i=N_{\alpha}+1}^{N_{\beta}} h_{ni} &= \sum_{i=1}^N I\{\xi_{\alpha}^{+} \leq h_i \leq \xi_{\beta}^{-}\} h_i + \xi_{\alpha}^{+}(\dot{N}_{\alpha}^{+} - N_{\alpha}) - \xi_{\beta}^{-}(N_{\beta}^{-} - N_{\beta}) \\ &\quad - \Delta\xi_{\alpha}I\{N_{\alpha} < \dot{N}_{\alpha}^{+}\}(\dot{N}_{\alpha}^{+} - N_{\alpha}) - \Delta\xi_{\beta}I\{N_{\beta}^{-} < N_{\beta}\}(N_{\beta}^{-} - N_{\beta}) \\ &\quad + \mathbb{L}_{\alpha} + \mathbb{L}_{\beta},\end{aligned}\tag{8}$$

where $\mathbb{L}_{\alpha} = J_{\alpha} - \bar{J}_{\alpha}$ with

$$J_{\alpha} = I\{N_{\alpha} < \dot{N}_{\alpha}^{+}\} \sum_{i=N_{\alpha}+1}^{\dot{N}_{\alpha}^{+}} (h_{ni} - \xi_{\alpha}^{-}), \quad \bar{J}_{\alpha} = I\{\dot{N}_{\alpha}^{+} \leq N_{\alpha}\} \sum_{i=\dot{N}_{\alpha}^{+}+1}^{N_{\alpha}} (h_{ni} - \xi_{\alpha}^{+})$$

and $\mathbb{L}_\beta = \bar{J}_\beta - J_\beta$ with

$$J_\beta = I\{N_\beta < N_\beta^-\} \sum_{i=N_\beta+1}^{N_\beta^-} (h_{ni} - \xi_\beta^-),$$

$$\bar{J}_\beta = I\{N_\beta^- \leq N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (h_{ni} - \xi_\beta^+).$$

Proof. For $i = 1, \dots, N$, write

$$\dot{h}_{ni} = (h_{ni} + \Delta\xi_\alpha)I\{h_{ni} < \xi_\alpha^+\} + h_{ni}I\{\xi_\alpha^+ \leq h_{ni} \leq \xi_\beta^-\} + (h_{ni} - \Delta\xi_\beta)I\{h_{ni} > \xi_\beta^-\}.$$

Since $\dot{h}_{ni} = h_{ni} + \Delta\xi_\alpha I\{h_{ni} < \xi_\alpha^+\} - \Delta\xi_\beta I\{h_{ni} > \xi_\beta^-\}$, $I\{h_{ni} < \xi_\alpha^+\} = I\{i \leq \dot{N}_\alpha^+\}$ and, by (6), $I\{h_{ni} > \xi_\beta^-\} = I\{i > N_\beta^-\}$, we can write

$$\begin{aligned} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni} &= \sum_{i=N_\alpha+1}^{N_\beta} \dot{h}_{ni} - \Delta\xi_\alpha I\{N_\alpha < \dot{N}_\alpha^+\}(\dot{N}_\alpha^+ - N_\alpha) \\ &\quad - \Delta\xi_\beta I\{N_\beta^- < N_\beta\}(N_\beta^- - N_\beta). \end{aligned} \quad (9)$$

Note that $h_{n\dot{N}_\alpha^+} < \xi_\alpha^+ \leq h_{n,\dot{N}_\alpha^++1}$ and $h_{nN_\beta^-} \leq \xi_\beta^- < h_{n,N_\beta^-+1}$. From (6), we have $I\{\xi_\alpha^+ \leq h_{ni} \leq \xi_\beta^-\} = I\{\dot{N}_\alpha^+ < i \leq N_\beta^-\}$. Hence, in (9),

$$\begin{aligned} \sum_{i=N_\alpha+1}^{N_\beta} \dot{h}_{ni} &= \sum_{i=\dot{N}_\alpha^++1}^{N_\beta^-} h_{ni} - I\{\dot{N}_\alpha^+ \leq N_\alpha\} \sum_{i=\dot{N}_\alpha^++1}^{N_\alpha} h_{ni} \\ &\quad + I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} + \Delta\xi_\alpha) - I\{N_\beta < N_\beta^-\} \sum_{i=N_\beta+1}^{N_\beta^-} h_{ni} \\ &\quad + I\{N_\beta^- \leq N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (h_{ni} - \Delta\xi_\beta) \\ &= \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i + \xi_\alpha^+ (\dot{N}_\alpha^+ - N_\alpha) - \xi_\beta^- (N_\beta^- - N_\beta) \\ &\quad + \mathbb{L}_\alpha + \mathbb{L}_\beta. \end{aligned} \quad (10)$$

Equation (8) follows from (9) and (10). This proves Lemma 2.1. \square

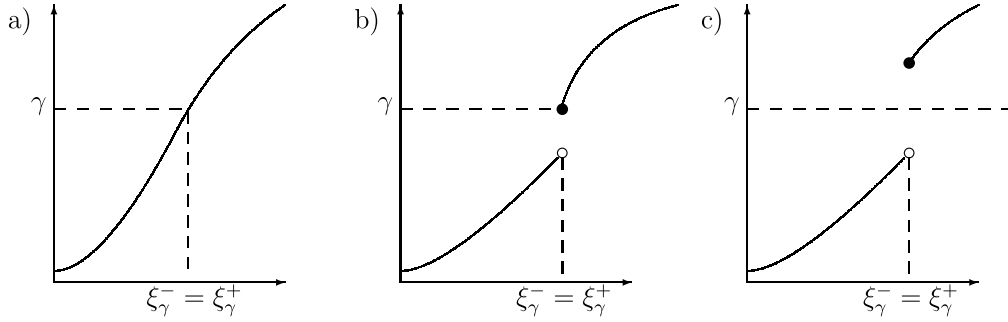


Figure 1. Plots of $H(\cdot)$ with $\xi_\gamma^- = \xi_\gamma^+$: (a) $H(\xi_\gamma^\pm) = \gamma = H(\xi_\gamma^\pm)$; (b) $H(\xi_\gamma^-) < \gamma = H(\xi_\gamma^+)$; (c) $H(\xi_\gamma^-) < \gamma < H(\xi_\gamma^+)$.

Lemma 2.2. *Note that*

$$\begin{aligned} N^{-1} \sum_{i=N_\alpha+1}^{N_\beta} h_{ni} &= N^{-1} \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i + \xi_\alpha^+ (H_n(\xi_\alpha^+) - \alpha) \\ &\quad - \xi_\beta^- (H_n(\xi_\beta^-) - \beta) - \Delta \xi_\alpha I\{N_\alpha < \dot{N}_\alpha^+\} (H_n(\xi_\alpha^+) - \alpha) \\ &\quad - \Delta \xi_\beta I\{N_\beta^- < N_\beta\} (H_n(\xi_\beta^-) - \beta) + n^{-1/2} \varrho_n, \end{aligned}$$

where $\varrho_n \rightarrow 0$ in probability as $n \rightarrow \infty$.

Proof. We shall estimate \mathbb{L}_α and \mathbb{L}_β , taking into account the values of the distribution function $H_F(x)$ at $x = \xi_\gamma^\pm$ with $\gamma = \alpha, \beta$. Figures 1 and 2 illustrate the different situations that need to be considered.

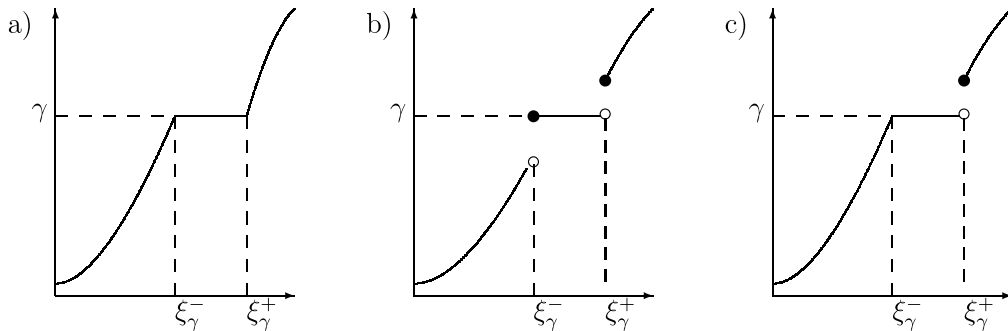


Figure 2. Plots of $H(\cdot)$ with $\xi_\gamma^- < \xi_\gamma^+$: (a) $\gamma = H(\xi_\gamma^-) = H(\xi_\gamma^+)$; (b) $H(\xi_\gamma^-) < \gamma = H(\xi_\gamma^-) = H(\xi_\gamma^+) < H(\xi_\gamma^+)$; (c) $H(\xi_\gamma^-) = \gamma = H(\xi_\gamma^+) < H(\xi_\gamma^+)$.

Estimating \mathbb{L}_α . Noting that $I\{\xi_\alpha^+ > h_{ni}\} = I\{i \leq \dot{N}_\alpha^+\}$, we write

$$\begin{aligned} J_\alpha &= I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} - \xi_\alpha^-) I\{\xi_\alpha^+ > h_{ni}\} \\ &= I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (h_{ni} - \xi_\alpha^-) I\{\xi_\alpha^- < h_{ni} < \xi_\alpha^+\} I\{N_\alpha^- < i \leq \dot{N}_\alpha^+\} \\ &\quad - I\{N_\alpha < \dot{N}_\alpha^+\} \sum_{i=N_\alpha+1}^{\dot{N}_\alpha^+} (\xi_\alpha^- - h_{ni}) I\{\xi_\alpha^- \geq h_{ni}\} I\{i \leq N_\alpha^-\} \\ &= J_\alpha^+ - J_\alpha^-. \end{aligned}$$

It is clear that if $\xi_\alpha^- = \xi_\alpha^+$, then $J_\alpha^+ = 0$ a.s. Let $\xi_\alpha^- \neq \xi_\alpha^+$, as is the case in Fig. 2. In this case, $H_F(\xi_\alpha^-) = \alpha = H_F(\xi_\alpha^+ -)$ and we can write

$$\begin{aligned} 0 \leq J_\alpha^+ &\leq I\{N_\alpha^- < \dot{N}_\alpha^+\} (\xi_\alpha^+ - \xi_\alpha^-) \sum_{i=N_\alpha^-+1}^{\dot{N}_\alpha^+} I\{\xi_\alpha^- < h_{ni} < \xi_\alpha^+\} \\ &\leq \Delta \xi_\alpha \sum_{i=1}^N I\{\xi_\alpha^- < h_i < \xi_\alpha^+\} = 0 \quad \text{a.s.} \end{aligned}$$

since $EI\{\xi_\alpha^- < h_i < \xi_\alpha^+\} = H_F(\xi_\alpha^+ -) - H_F(\xi_\alpha^-) = 0$. Hence, we always have the relation $J_\alpha^+ = 0$ a.s. To estimate J_α^- , we note first that if $\dot{N}_\alpha^+ > N_\alpha^-$, then the indicator $I\{i \leq N_\alpha^-\} = 0$ for all $i = N_\alpha^- + 1, \dots, \dot{N}_\alpha^+$. Therefore, we have the inequalities

$$\begin{aligned} 0 \leq J_\alpha^- &\leq I\{N_\alpha < N_\alpha^-\} \sum_{i=N_\alpha+1}^{N_\alpha^-} (\xi_\alpha^- - h_{ni}) I\{\xi_\alpha^- \geq h_{ni}\} \\ &\leq I\{N_\alpha < N_\alpha^-\} (N_\alpha^- - N_\alpha) (\xi_\alpha^- - h_{nN_\alpha}) I\{\xi_\alpha^- \geq h_{nN_\alpha}\}. \end{aligned} \quad (11)$$

Further, we shall apply the technique used in Smirnov [22] with a probability inequality from Hoeffding [14] (or see, for example, Serfling [17], pages 75 and 201). Thus,

$$\begin{aligned} &P\{(\xi_\alpha^- - h_{nN_\alpha} > \varepsilon) \cap (\xi_\alpha^- \geq h_{nN_\alpha})\} \\ &\leq P\{\xi_\alpha^- - h_{nN_\alpha} \geq \varepsilon\} \\ &= P\{H_n(\xi_\alpha^- - \varepsilon) \geq N^{-1}N_\alpha\} \\ &= P\{H_n(\xi_\alpha^- - \varepsilon) - H(\xi_\alpha^- - \varepsilon) \geq N^{-1}N_\alpha - H(\xi_\alpha^- - \varepsilon)\} \\ &\leq c_1 \exp\{-c_2 n \theta_\alpha^2(\xi_\alpha^-, \varepsilon)\} \end{aligned} \quad (12)$$

with some positive constants c_1 and c_2 , depending only on m and $\theta_\alpha(\xi_\alpha^-, \varepsilon) = \alpha - H_F(\xi_\alpha^- - \varepsilon)$. Further, $\theta_\alpha(\xi_\alpha^-, \varepsilon) > 0$ for any small values of $\varepsilon > 0$, by the definition of the smallest α -quantile ξ_α^- . Under the conditions of the lemma, $\sqrt{n}N^{-1}(N_\alpha^- - N_\alpha) \xrightarrow{d} \tau_\alpha^-$ as $n \rightarrow \infty$. Hence, $\sqrt{n}N^{-1}J_\alpha \rightarrow 0$ in probability as $n \rightarrow \infty$.

Next, we consider \bar{J}_α . By definition, $\dot{N}_\alpha^+ \leq N_\alpha^+$ and since $I\{h_{ni} < \xi_\alpha^+\} = I\{i \leq \dot{N}_\alpha^+\}$ and $h_{nN_\alpha^+} \leq \xi_\alpha^+ < h_{n, N_\alpha^++1}$, it follows that the indicator $I\{h_{ni} = \xi_\alpha^+\} = 1$ for $i = \dot{N}_\alpha^+ + 1, \dots, N_\alpha^+$ and we can write

$$\begin{aligned} 0 \leq \bar{J}_\alpha &= I\{\dot{N}_\alpha^+ \leq N_\alpha\} \sum_{i=\dot{N}_\alpha^++1}^{N_\alpha} (h_{ni} - \xi_\alpha^+) I\{h_{ni} \geq \xi_\alpha^+\} \\ &= I\{N_\alpha^+ \leq N_\alpha\} \sum_{i=N_\alpha^++1}^{N_\alpha} (h_{ni} - \xi_\alpha^+) I\{h_{ni} > \xi_\alpha^+\} \\ &\leq I\{N_\alpha^+ \leq N_\alpha\} (N_\alpha - N_\alpha^+) (h_{nN_\alpha} - \xi_\alpha^+) I\{h_{nN_\alpha} > \xi_\alpha^+\}. \end{aligned} \quad (13)$$

In (13), we need to consider two cases: $H_F(\xi_\alpha^+) = \alpha$ and $\alpha < H_F(\xi_\alpha^+)$. In the first case, $H_F(\xi_\alpha^+) = \alpha$ and we have the weak convergence $\sqrt{n}N^{-1}(N_\alpha^+ - N_\alpha) \xrightarrow{d} \tau_\alpha^+$ as $n \rightarrow \infty$ and the following estimates which are similar to (12):

$$\begin{aligned} &P\{(h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon) \cap (h_{nN_\alpha} > \xi_\alpha^-)\} \\ &\leq P\{h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon\} \\ &= P\{N^{-1}N_\alpha > H_n(\xi_\alpha^+ + \varepsilon)\} \\ &= P\{H(\xi_\alpha^+ + \varepsilon) - H_n(\xi_\alpha^+ + \varepsilon) > H(\xi_\alpha^+ + \varepsilon) - N^{-1}N_\alpha\} \\ &\leq c_1 \exp\{-c_2 n \delta_\alpha^2(\xi_\alpha^+, \varepsilon)\}, \end{aligned} \quad (14)$$

where $\delta_\alpha(\xi_\alpha^+, \varepsilon) = H_F(\xi_\alpha^+ + \varepsilon) - \alpha$. In addition, $\delta_\alpha(\xi_\alpha^+, \varepsilon) > 0$ for any small values of $\varepsilon > 0$ because of the definition of the largest α -quantile ξ_α^+ . Hence, in the case $H_F(\xi_\alpha^+) = \alpha$, we have $\sqrt{n}N^{-1}\bar{J}_\alpha \rightarrow 0$ in probability as $n \rightarrow \infty$. In the second case in (13), $\delta_\alpha(\xi_\alpha^+, 0) = H_F(\xi_\alpha^+) - \alpha > 0$ and we have the representation

$$\sqrt{n}N^{-1}(N_\alpha^+ - N_\alpha) = \sqrt{n}(H_n(\xi_\alpha^+) - H_F(\xi_\alpha^+)) + \sqrt{n}\delta_\alpha(\xi_\alpha^+, 0) + \omega_n(\alpha), \quad (15)$$

where $\sqrt{n}(H_n(\xi_\alpha^+) - H_F(\xi_\alpha^+)) \xrightarrow{d} \tau_\alpha^+$ and $\omega_n(\alpha) = \sqrt{n}N^{-1}(\alpha N - [\alpha N]) = O(n^{-1/2})$ as $n \rightarrow \infty$, but the positive term $\sqrt{n}\delta_\alpha(\xi_\alpha^+, 0)$ is unbounded. Therefore, in this case, we shall apply the estimate (14) with εn^{-1} instead of ε , that is, $P\{(h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon n^{-1}) \cap (h_{nN_\alpha} > \xi_\alpha^-)\} \leq c_1 \exp\{-c_2 n \delta_\alpha^2(\xi_\alpha^+, \varepsilon n^{-1})\}$. Since the distribution function H_F is continuous from the right at the point ξ_α^+ it follows that $\delta_\alpha(\xi_\alpha^+, 0) \leq \delta_\alpha(\xi_\alpha^+, \varepsilon n^{-1})$ for any small $\varepsilon > 0$ and sufficiently large n . Hence, in the second case, $\alpha < H_F(\xi_\alpha^+)$ and (14) is replaced by the inequality

$$P\{(h_{nN_\alpha} - \xi_\alpha^+ > \varepsilon n^{-1}) \cap (h_{nN_\alpha} > \xi_\alpha^-)\} \leq c_1 \exp\{-c_2 n \delta_\alpha^2(\xi_\alpha^+, 0)\}, \quad (16)$$

which provides the desired convergence $\sqrt{n}N^{-1}\bar{J}_\alpha \rightarrow 0$ in probability as $n \rightarrow \infty$. Thus, we have proven that $\sqrt{n}N^{-1}\mathbb{L}_\alpha \rightarrow 0$ in probability as $n \rightarrow \infty$.

Estimating \mathbb{L}_β . Noting that $I\{\xi_\beta^- \geq h_{ni}\} = I\{i \leq N_\beta^-\}$, we write

$$\begin{aligned} 0 \leq -J_\beta &= I\{N_\beta < N_\beta^-\} \sum_{i=N_\beta+1}^{N_\beta^-} (\xi_\beta^- - h_{ni}) I\{\xi_\beta^- \geq h_{ni}\} \\ &\leq I\{N_\beta < N_\beta^-\} (N_\beta^- - N_\beta) (\xi_\beta^- - h_{nN_\beta}) I\{\xi_\beta^- \geq h_{nN_\beta}\}. \end{aligned}$$

Here, by analogy with (12), we have

$$\begin{aligned} P\{(\xi_\beta^- - h_{nN_\beta} > \varepsilon) \cap (\xi_\beta^- \geq h_{nN_\beta})\} &\leq P\{\xi_\beta^- - h_{nN_\beta} \geq \varepsilon\} \\ &= P\{H_n(\xi_\beta^- - \varepsilon) \geq N^{-1}N_\beta\} \\ &= P\{H_n(\xi_\beta^- - \varepsilon) - H(\xi_\beta^- - \varepsilon) \geq N^{-1}N_\beta - H(\xi_\beta^- - \varepsilon)\} \\ &\leq c_1 \exp\{-c_2 n \theta_\beta^2(\xi_\beta^-, \varepsilon)\} \end{aligned} \quad (17)$$

with $\theta_\beta(\xi_\beta^-, \varepsilon) = \beta - H_F(\xi_\beta^- - \varepsilon)$ and by analogy with (15),

$$\sqrt{n}N^{-1}(N_\beta^- - N_\beta) = \sqrt{n}(H_n(\xi_\beta^-) - H_F(\xi_\beta^-)) + \sqrt{n}\theta_\beta(\xi_\beta^-, 0) + \omega_n(\beta). \quad (18)$$

Here, we need to consider two cases: $\beta - H_F(\xi_\beta^-) = 0$ and $\beta - H_F(\xi_\beta^-) > 0$. In the first case, we apply the inequality (17) with sufficiently small $\varepsilon > 0$. In the second case, we use (17) again, but with parameter εn^{-1} , as in (16), to get

$$P\{(\xi_\beta^- - h_{nN_\beta} > \varepsilon n^{-1}) \cap (\xi_\beta^- \geq h_{nN_\beta})\} \leq c_1 \exp\{-c_2 n \theta_\beta^2(\xi_\beta^-, 0)\} \quad (19)$$

since the distribution function H_F has a limit from the left at the point ξ_β^- and $H_F(\xi_\beta^-) \geq H_F(\xi_\beta^- - \varepsilon n^{-1})$. In this result, we have $\sqrt{n}N^{-1}J_\beta \rightarrow 0$ in probability as $n \rightarrow \infty$.

Finally, we consider \bar{J}_β . Since $I\{h_{ni} > \xi_\beta^\pm\} = I\{i > N_\beta^\pm\}$, we write

$$\begin{aligned} \bar{J}_\beta &= I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (h_{ni} - \xi_\beta^+) I\{h_{ni} > \xi_\beta^-\} \\ &= -I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta^-+1}^{N_\beta} (\xi_\beta^+ - h_{ni}) I\{\xi_\beta^- < h_{ni} < \xi_\beta^+\} I\{N_\beta^- < i \leq \dot{N}_\beta^+\} \\ &\quad + I\{N_\beta^- < N_\beta\} \sum_{i=N_\beta^++1}^{N_\beta} (h_{ni} - \xi_\beta^+) I\{h_{ni} > \xi_\beta^+\} I\{i > \dot{N}_\beta^+\} \\ &= -\bar{J}_\beta^- + \bar{J}_\beta^+. \end{aligned}$$

If $\xi_\beta^- = \xi_\beta^+$, then $\bar{J}_\beta^- = 0$ a.s. Now, assume that $\xi_\beta^- \neq \xi_\beta^+$. In this case, $H_F(\xi_\beta^-) = \beta = H_F(\xi_\beta^+ -)$ and we have

$$\begin{aligned} 0 \leq \bar{J}_\beta^- &\leq I\{N_\beta^- < \dot{N}_\beta^+\}(\xi_\beta^+ - \xi_\beta^-) \sum_{i=N_\beta^-+1}^{\dot{N}_\beta^+} I\{\xi_\beta^- < h_{ni} < \xi_\beta^+\} \\ &\leq \Delta\xi_\beta \sum_{i=1}^N I\{\xi_\beta^- < h_i < \xi_\beta^+\} = 0 \quad \text{a.s.} \end{aligned}$$

since $EI\{\xi_\beta^- < h_i < \xi_\beta^+\} = H_F(\xi_\beta^+ -) - H_F(\xi_\beta^-) = 0$. Hence, we always have $\bar{J}_\beta^- = 0$ a.s. To estimate \bar{J}_β^+ , we write

$$0 \leq \bar{J}_\beta^+ \leq I\{N_\beta^- < N_\beta\}(N_\beta^+ - N_\beta)(h_{ni} - \xi_\beta^+)I\{h_{ni} > \xi_\beta^+\}$$

and apply the estimates (13)–(16) with β instead of α . We have $\sqrt{n}N^{-1}\bar{J}_\beta \rightarrow 0$ in probability as $n \rightarrow \infty$ and hence $\sqrt{n}N^{-1}\mathbb{L}_\beta \rightarrow 0$ in probability as $n \rightarrow \infty$.

This proves Lemma 2.2. \square

Proof of Theorem 1.1. Let $U(g)$ be a U -statistic of the form (1) with the kernel

$$\begin{aligned} g(x_1, \dots, x_m) &= [I\{h(x_1, \dots, x_m) \leq \xi_\beta^-\}(h(x_1, \dots, x_m) - \xi_\beta^-) + \beta\xi_\beta^-] \\ &\quad - [I\{h(x_1, \dots, x_m) < \xi_\alpha^+\}(h(x_1, \dots, x_m) - \xi_\alpha^+) + \alpha\xi_\alpha^+]. \end{aligned}$$

We see that

$$U(g) = N^{-1} \sum_{i=1}^N I\{\xi_\alpha^+ \leq h_i \leq \xi_\beta^-\} h_i + \xi_\alpha^+ (H_n(\xi_\alpha^+ -) - \alpha) - \xi_\beta^- (H_n(\xi_\beta^-) - \beta).$$

It is not difficult to verify for this function that $Eg(X_1, \dots, X_m) = \theta$ and $g(x) = Eg(x, X_2, \dots, X_m) - \theta, x \in \mathbb{X}$; in addition, $Eg^2(X) > 0$, by the condition of the theorem. Hence, the kernel g is non-degenerate and, by the central limit theorem for U -statistics with such bounded kernels, we have the weak convergence $\tau_{ng} := m^{-1}\sqrt{n}(U(g) - \theta) \xrightarrow{d} \tau_g$ as $n \rightarrow \infty$ (see, for example, Borovskikh [3]). By the same central limit theorem, we have

$$\tau_{n\alpha} := m^{-1}\sqrt{n}(H_n(\xi_\alpha^+ -) - H(\xi_\alpha^+ -)) \xrightarrow{d} \tau_\alpha$$

and

$$\tau_{n\beta} := m^{-1}\sqrt{n}(H_n(\xi_\beta^-) - H(\xi_\beta^-)) \xrightarrow{d} \tau_\beta$$

as $n \rightarrow \infty$. Under the conditions of the theorem, we have $E|I\{\dot{N}_\alpha^+ - N_\alpha > 0\} - I\{\tau_\alpha > 0\}| \rightarrow 0$ if $\Delta\xi_\alpha \neq 0$ (in this case, $H(\xi_\alpha^+ -) = \alpha$) and $E|I\{N_\beta^- - N_\beta < 0\} - I\{\tau_\beta < 0\}| \rightarrow 0$

if $\Delta\xi_\beta \neq 0$ (in this case, $H(\xi_\beta^-) = \beta$). Further, it is easy to prove that the covariances $\text{Cov}(\tau_{n*}, \tau_{n*}) \rightarrow \text{Cov}(\tau_*, \tau_*)$ as $n \rightarrow \infty$, where $*, * = \alpha, g, \beta$. Now, apply Lemma 2.2 to complete the proof of Theorem 1.1. \square

Lemma 2.3. *The following representation holds:*

$$L_{\alpha\beta} = N^{-1} \sum_{i=\bar{N}_\alpha}^{\bar{N}_\beta-1} h_{ni}.$$

Proof. By definition, we can write

$$\begin{aligned} L_{\alpha\beta} &= \int_R I\{h_\alpha \leq x < h_\beta\} x dH_n(x) \\ &= \frac{1}{N} \sum_{i=1}^N I\{h_\alpha \leq h_{ni} < h_\beta\} h_{ni} \\ &= \frac{1}{N} \sum_{i=1}^N I\{h_{ni} < h_\beta\} h_{ni} - \frac{1}{N} \sum_{i=1}^N I\{h_{ni} < h_\alpha\} h_{ni} \\ &= \frac{1}{N} \sum_{i=1}^{\bar{N}_\beta-1} h_{ni} - \frac{1}{N} \sum_{i=1}^{\bar{N}_\alpha-1} h_{ni} \\ &= \frac{1}{N} \sum_{i=\bar{N}_\alpha}^{\bar{N}_\beta-1} h_{ni}. \end{aligned}$$

This proves Lemma 2.3. \square

The proof of Corollary 1.2 follows from Theorem 1.1 and Lemma 2.3.

References

- [1] Akritas, M.G. (1986). Empirical processes associated with V -statistics and a class of estimates under random censoring. *Ann. Statist.* **14** 619–637. [MR0840518](#)
- [2] Bickel, P.J. (1967). Some contributions to the theory of order statistics. In *Proc. Fifth Berkeley Symp. Math. Stat. Probab.* **1** 575–591. Berkeley, CA: California Univ. Press. [MR0216701](#)
- [3] Borovskikh, Y.V. (1996). *U-Statistics in Banach Spaces*. Utrecht, The Netherlands: VSP. [MR1419498](#)
- [4] Borovskikh, Y.V. and Weber, N.C. (2008). Asymptotic distributions of non-degenerate U -statistics on trimmed samples. *Statist. Probab. Lett.* **78** 336–346. [MR2396407](#)
- [5] Cheng, S. (1992). A complete solution for weak convergence of heavily trimmed sums. *Sci. China Ser. A* **35** 641–656. [MR1196624](#)

- [6] Chernoff, H., Gastwirth, J. and Johns, M. (1967). Asymptotic distributions of linear combinations of functions of order statistics with application to estimation. *Ann. Math. Statist.* **38** 52–72. [MR0203874](#)
- [7] Choudhury, J. and Serfling, R.J. (1988). Generalized order statistics, Bahadur representations and sequential nonparametric fixed-width confidence intervals. *J. Statist. Plann. Inference* **19** 269–282. [MR0955393](#)
- [8] Csörgö, S., Haeusler, E. and Mason, D.M. (1988). The asymptotic distribution of trimmed sums. *Ann. Probab.* **16** 672–699. [MR0929070](#)
- [9] Goldie, C.M. (1977). Convergence theorems for empirical Lorenz curves and their inverses. *Adv. in Appl. Probab.* **9** 765–791. [MR0478267](#)
- [10] Gijbels, I., Janssen, P. and Veraverbeke, N. (1988). Weak and strong representations for trimmed U -statistics. *Probab. Theory Related Fields* **77** 179–194. [MR0927236](#)
- [11] Griffin, P.S. and Pruitt, W.E. (1989). Asymptotic normality and subsequential limits of trimmed sums. *Ann. Probab.* **17** 1186–1219. [MR1009452](#)
- [12] Helmers, R. and Ruymgaart, F.H. (1988). Asymptotic normality of generalized L -statistics with unbounded scores. *J. Statist. Plann. Inference* **19** 43–53. [MR0944195](#)
- [13] Helmers, R. and Zitikis, R. (2005). Strong laws for generalized absolute Lorenz curves when data are stationary and ergodic sequences. *Proc. Amer. Math. Soc.* **133** 3703–3712. [MR2163610](#)
- [14] Hössjer, O. (1996). Incomplete generalized L -statistics. *Ann. Statist.* **24** 2631–2654. [MR1425972](#)
- [15] Janssen, P., Serfling, R. and Veraverbeke, N. (1984). Asymptotic normality for a general class of statistical functions and applications to measures of spread. *Ann. Statist.* **12** 1369–1379. [MR0760694](#)
- [16] Mason, D. and Shorack, G. (1992). Necessary and sufficient conditions for asymptotic normality of L -statistics. *Ann. Probab.* **20** 1779–1804. [MR1188042](#)
- [17] Serfling, R.J. (1980). *Approximation Theorems of Mathematical Statistics*. New York: Wiley. [MR0595165](#)
- [18] Serfling, R.J. (1984). Generalized L -, M - and R -statistics. *Ann. Statist.* **12** 76–86. [MR0733500](#)
- [19] Shorack, G.R. (1972). Functions of order statistics. *Ann. Math. Statist.* **43** 412–427. [MR0301846](#)
- [20] Shorack, G.R. (1974). Random means. *Ann. Statist.* **2** 661–675. [MR0415887](#)
- [21] Silverman, B.W. (1983). Convergence of a class of empirical distribution functions of dependent random variables. *Ann. Probab.* **11** 745–751. [MR0704560](#)
- [22] Smirnov, N.V. (1970). *Probability Theory and Mathematical Statistics. Selected Works* 51–58. Moscow: Nauka. [MR0265117](#)
- [23] Stigler, S.M. (1973). The asymptotic distribution of the trimmed mean. *Ann. Statist.* **1** 472–477. [MR0359134](#)
- [24] Stigler, S.M. (1974). Linear functions of order statistics with smooth weight functions. *Ann. Statist.* **2** 676–693. [MR0373152](#)

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